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# Spectral properties of some Jacobi matrices with double weights

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**Abstract**

We study the spectral properties of Jacobi matrices with the weights satisfying $\lambda_{2n-1} = \lambda_{2n} = n^a$ or $\lambda_{2n} = \lambda_{2n+1} = n^a$, $a > 0$. We show that for $a = 1$ these are cases of spectral phase transitions in $a$. We use a new method of estimating transfer matrix products to describe the absolutely continuous part of these operators. For $a = 1$ the existence of a spectral gap is proved. We also show how the results for double weights can be used for the spectral analysis of the Jacobi matrices related to some birth and death processes, previously studied by Janas and Naboko.


## 0. Introduction

Operator problems for Jacobi matrices with unbounded double weights have already appeared in literature. For instance, the significant self-adjointness problem was studied in [5,15], some spectral problems were analyzed in [5,6,16]. This kind of Jacobi matrices is important also for applications in physics; see, e.g., [7].

So far, the spectral studies have been based on commutator methods and orthogonal polynomials theory. The aim of this paper is to show the usefulness of other methods. They are mainly based on Khan and Pearson subordination theory, which has recently proved to be a strong tool for spectral analysis of Jacobi operators. To use this theory, it is convenient to have some estimates for the norms of products of large numbers of 2 by 2 matrices (see, e.g., [8,9,18]). Such estimates are in general easier to obtain when these matrices converge

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to a limit having distinct proper values. Here we show a general method working for some cases when the proper values coincide.

Our purpose is also to show that our results for double weights can be applied to study some other Jacobi matrices, related to birth and death processes.

Let us set up here the necessary notation and terminology. In this paper Jacobi matrix is a maximal operator $J$ in $l^2(\mathbb{N})$ (here $\mathbb{N} = \{1, 2, 3, \ldots\}$) defined by the infinite tridiagonal matrix of the form

$$\begin{pmatrix} q_1 & \lambda_1 & & & \\ \lambda_1 & q_2 & \lambda_2 & & \\ & \lambda_2 & q_3 & \lambda_3 & \\ & & \lambda_3 & q_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

More precisely, for a complex sequence $u = \{u_n\}$ and $n \in \mathbb{N}$ we define

$$(\mathcal{J}u)_n := \lambda_{n-1}u_{n-1} + q_n u_n + \lambda_n u_{n+1} \tag{0.1}$$

with the convention that $\lambda_j = u_j := 0$, if $j < 1$, the domain $D(J)$ of $J$ is given by $D(J) := \{u \in l^2(\mathbb{N})\colon \mathcal{J}u \in l^2(\mathbb{N})\}$ and $Ju = \mathcal{J}u$ for $u \in D(J)$. For a given Jacobi matrix the sequences $\{\lambda_n\}$ (*the weight sequence*) and $\{q_n\}$ (*the diagonal sequence*) always denote the sequences determined by (0.1), and we assume here that $\lambda_n, q_n \in \mathbb{R}$, $\lambda_n \neq 0$ for $n \in \mathbb{N}$.

We consider two special classes of *Jacobi matrices with double weights*. For a positive sequence $\alpha = \{\alpha_n\}$ and $\epsilon > 0$ we define $J_\alpha$ to be the Jacobi matrix with $\lambda_{2n-1} = \lambda_{2n} = \alpha_n$, $q_n = 0$, for $n \in \mathbb{N}$ (i.e., $\{\lambda_n\} = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots)$) and we define $J_{\epsilon,\alpha}$ to be the Jacobi matrix with $\lambda_1 = \epsilon$, $\lambda_{2n} = \lambda_{2n+1} = \alpha_n$, $q_n = 0$ for $n \in \mathbb{N}$ (i.e., $\{\lambda_n\} = (\epsilon, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots)$). In [15] and also in [5] it was proved that for any choice of $\alpha$ and $\epsilon$ the operators $J_\alpha$ and $J_{\epsilon,\alpha}$ are self-adjoint.

For a self-adjoint operator $A$ in a Hilbert space let $A_{\mathrm{ac}}$ be the absolutely continuous part of $A$, let $E_A$ be the projection valued spectral measure for $A$ and for a Borel subset $G$ of $\mathbb{R}$ let $(A)_G$ denote the restriction of $A$ to the range of $E_A(G)$. We call $A$ *absolutely continuous* if $A = A_{\mathrm{ac}}$ (the name "$\sigma(A)$ is purely absolutely continuous" is often used in literature in this case); we call $A$ *absolutely continuous in* $G$ if $(A)_G$ is absolutely continuous. Moreover, $A$ is *discrete* if $\sigma_{\mathrm{ess}}(A) = \emptyset$ (the name "$\sigma(A)$ is purely discrete" is also frequently used) and $A$ is *discrete in* $G$ if $\sigma_{\mathrm{ess}}(A) \cap G = \emptyset$. The domain of $A$ we denote by $D(A)$.

In Section 1 we introduce the notion of $H$*-class* of matrix sequences to describe a kind of behaviour of transfer matrices for Jacobi matrix $J$, which can be used to find some regions of the absolute continuity of $J$. The results of this section are a compilation of some previously known results—the subordination theory of Khan and Pearson [13] and some method of Behncke [1] and Stolz [18] generalized by Janas and Naboko [9].

In Section 2 we formulate two criterions for $H$-class. The first has been already used (maybe not explicitly) in some papers (see, e.g., [8]). The second seems to be in fact a new tool, and it is particularly convenient for the case of Jacobi matrices with double weights.

The main results are presented in Section 3, where we study the spectral properties of $J_\alpha$ and $J_{\epsilon,\alpha}$ with $\alpha_n = n^a$. We prove that for $0 < a < 1$, $J_\alpha$ is absolutely continuous and $\sigma_{\mathrm{ac}}(J_\alpha) = \mathbb{R}$, for $a > 1$, $J_\alpha$ is discrete, and that for $a = 1$, $J_\alpha$ has a gap in the spectrum

($\sigma(J_\alpha) = \mathbb{R} \setminus (-1/2; 1/2)$). The case of $J_{\epsilon,\alpha}$ is similar, but slightly more complicated. The proofs are based on our results of previous sections, but to prove the presence of the spectral gap we compute also the explicit form of $J_\alpha^{-1}$. It is worth to note that also the Hardy inequality for the Cesaro operator plays an important role here.

The case $a = 1$ was considered first by Dombrowski and Pedersen in [5] (see also [16] for some generalizations), where the absolute continuity was proved, but the commutator methods used there do not enable to compute the spectrum. The existence of the spectral gap seems to be interesting, since the spectra of the most of Jacobi matrices with unbounded weights having been computed before are very simple—usually they are discrete or equal to $\mathbb{R}$, or to a half line with a possible discrete component (see, e.g., [6,8–10,12]).

In Section 4 we show some connections of our results with the Jacobi matrices related to some birth and death process (see [14]). The detailed spectral analysis of these operators were presented by Janas and Naboko in [12]. We show that some of their findings can also be obtained by the use of our results from Section 3.

## 1. Subordination theory and the $H$-class

For $\lambda \in \mathbb{R}$ consider the recurrent equation for a complex sequence $\{u_n\}$ (*the formal proper equation for $J$ and $\lambda$*)

$$(\mathcal{J}u)_n = \lambda u_n, \quad n \geqslant 2 \tag{1.1}$$

(note that here we omit $n = 1$). This is the second-order difference equation and the space of all the solutions is two-dimensional. Below we formulate a theorem being a conclusion of the Khan and Pearson subordination theory and the Janas and Naboko generalization of the so-called Behncke–Stolz lemma (this generalization we call later "GBS lemma"). The theorem shows a connection of some absolute continuity regions with the behaviour of the formal proper solutions.

Let us first introduce some more notations. For a Borel subset $G$ of $\mathbb{R}$ and for a Borel measure $\mu$ on $\mathbb{R}$ the restriction of $\mu$ to $G$ is denoted by $\mu_G$ and $\mathbf{X}_{G,\mu}$ denotes the operator of multiplication by the independent variable '$x$' in the space $L^2(G, d\mu_G)$ (with the usual maximal domain). We simplify this and write $\mathbf{X}_G$ when $\mu$ is the one-dimensional Lebesgue measure, denoted here by $dx$ or $|\cdot|$. The scalar valued spectral measure for a self-adjoint operator $A$ and for the vector $v$ we denote by $E_{A,v}$. The topological support of a measure is denoted by supp. The symbol $\sim$ denotes the unitary equivalence of operators. The canonical orthonormal base of $l^2(\mathbb{N})$ is denoted by $\{e_n\}$.

**Theorem 1.1.** *Suppose that $J$ is a self-adjoint Jacobi matrix and $G$ is an open subset of $\mathbb{R}$ such that for any $\lambda \in G$ and for any solution $u$ of* (1.1)

$$\exists M > 0\ \forall n \in \mathbb{N} \quad \sum_{k=1}^{n+1} |u_k|^2 \leqslant M \sum_{k=1}^{n} \frac{1}{|\lambda_k|}.$$

*Then $(J)_G \sim \mathbf{X}_G$ and consequently $J$ is absolutely continuous in $G$ and $\overline{G} \subset \sigma_{\mathrm{ac}}(J)$.*

**Proof.** Let $\mu = E_{J,e_1}$. It is easily seen that $e_1$ is a cyclic vector for $J$ (i.e., $J$ restricted to the space $\{f(J)e_1\colon\ f$ is a polynomial$\}$ is essentially self-adjoint). Thus, by the functional calculus theorem, $J \sim \mathbf{X}_{\mathbb{R},\mu}$ and $\sigma_{\mathrm{ac}}(J) = \operatorname{supp}\mu_{\mathrm{ac}}$, where $\mu_{\mathrm{ac}}$ is the absolute continuous part of $\mu$. By the definition of $(A)_G$ we thus get $(J)_G \sim (\mathbf{X}_{\mathbb{R},\mu})_G \sim \mathbf{X}_{G,\mu}$. Using GBS lemma (see [8, Lemma 1.5] for the rigorous formulation) we obtain the absence of subordinated solutions of (1.1) for all $\lambda \in G$. Hence, by Khan and Pearson theorem [13], we have

$$\forall \omega \in \mathcal{B}(G) \quad \mu(\omega) = 0 \quad \text{iff} \quad |\omega| = 0,$$

where $\mathcal{B}(G)$ is the family of Borel subsets of $G$. Therefore $\mu_G = f\,dx$ for a *strictly* positive function from $L^1(G, dx)$. It follows that the formula $U\varphi := \sqrt{f}\varphi$ defines a unitary transformation $U : L^2(G, \mu_G) \to L^2(G, dx)$, such that $U\mathbf{X}_{G,\mu}U^{-1} = \mathbf{X}_G$. Thus $(J)_G \sim \mathbf{X}_G$, which also shows that $J$ is absolutely continuous in $G$. Moreover, since $G$ is open, $\overline{G} = \sigma(\mathbf{X}_G) = \sigma_{\mathrm{ac}}(\mathbf{X}_G) = \sigma_{\mathrm{ac}}((J)_G) \subset \sigma_{\mathrm{ac}}(J)$. $\square$

Let us define now *the transfer matrices* $B_n(\lambda)$ for Jacobi matrix $J$ and $\lambda \in \mathbb{C}$ to be the $2 \times 2$ matrices satisfying

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = B_n(\lambda) \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \quad n \geqslant 2,$$

for any solution $\{u_n\}$ of (1.1). Thus we have

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{\lambda_{n-1}}{\lambda_n} & \frac{\lambda - q_n}{\lambda_n} \end{pmatrix}. \tag{1.2}$$

We introduce here a class of sequences of matrices—*the $H$-class*—which is very convenient to study the absolutely continuous part of $J$. Let $\|\cdot\|$ be the standard operator norm for matrices. We use the following convention for the product of matrices: $\prod_{k=m}^{n} A_k$ equals $A_n \ldots A_m$ if $n > m$; if $n = m$ it equals $A_m$; and if $n < m$ it equals $I$. The symbol $\{a_n\}_{n \geqslant n_0}$ is used to denote a sequence defined on $\{n_0, n_0+1, n_0+2, \ldots\}$ (and $\{a_n\} = \{a_n\}_{n \geqslant 1}$).

**Definition 1.1.** Consider a sequence $\{C_n\}_{n \geqslant n_0}$ of complex $2 \times 2$ matrices; $\{C_n\}_{n \geqslant n_0} \in H$ iff

$$\exists M > 0\ \forall n \geqslant n_0 \quad \left\| \prod_{k=n_0}^{n} C_k \right\|^2 \leqslant M \prod_{k=n_0}^{n} |\det C_k|.$$

Let us mention two easy-to-check properties of $H$-class which we use in further sections.

**Proposition 1.1.** *Let $\{C_n\}_{n \geqslant n_0}$ be a sequence of complex $2 \times 2$ invertible matrices.*

(a) *If $n_1 \geqslant n_0$, then $\{C_n\}_{n \geqslant n_0} \in H$ iff $\{C_n\}_{n \geqslant n_1} \in H$.*
(b) *If $\{C_n\}_{n \geqslant n_0}$ and $\{C_n^{-1}\}_{n \geqslant n_0}$ are bounded, then $\{C_n\}_{n \geqslant n_0} \in H$ iff $\{C_{2n}C_{2n-1}\}_{n \geqslant n_0} \in H$ iff $\{C_{2n+1}C_{2n}\}_{n \geqslant n_0} \in H$.*

Now, using the notion of $H$-class we can reformulate the previous theorem in terms of transfer matrices for $J$.

**Theorem 1.2.** *If $J$ is a self-adjoint Jacobi matrix and $G$ is an open subset of $\mathbb{R}$ such that $\{B_n(\lambda)\}_{n\geqslant 2} \in H$ for any $\lambda \in G$, then $(J)_G \sim \mathbf{X}_G$ and consequently $J$ is absolutely continuous in $G$ and $\overline{G} \subset \sigma_{\mathrm{ac}}(J)$.*

**Proof.** Let $u$ be a solution of (1.1) for $\lambda \in G$. For $k \geqslant 2$ we have

$$\begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} = \prod_{s=2}^{k} B_s(\lambda) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and thus, using $\{B_n(\lambda)\}_{n\geqslant 2} \in H$ and (1.2), for some $M'$ we obtain

$$\begin{aligned} |u_{k+1}|^2 &\leqslant \left\| \prod_{s=2}^{k} B_s(\lambda) \right\|^2 \left(|u_1|^2 + |u_2|^2\right) \\ &\leqslant M' \prod_{s=2}^{k} \left|\det B_s(\lambda)\right| = M' \prod_{s=2}^{k} \frac{|\lambda_{s-1}|}{|\lambda_s|} = \frac{M'|\lambda_1|}{|\lambda_k|}. \end{aligned}$$

Now, for $M''$ large enough

$$\sum_{k=1}^{n+1} |u_k|^2 = |u_1|^2 + |u_2|^2 + \sum_{k=2}^{n} |u_{k+1}|^2 \leqslant M'' \sum_{k=1}^{n} \frac{1}{|\lambda_k|}$$

for $n \geqslant 2$, and we can use Theorem 1.1. □

We end with a simple example illustrating the above theorem.

**Example 1.1.** Let $J$ be the free discrete Schrödinger operator, i.e., $\lambda_n = 1$ and $q_n = 0$ for each $n \in \mathbb{N}$. Then transfer matrices are $n$-independent, diagonalizable for $|\lambda| \neq 2$, and for $\lambda \in (-2; 2) =: G$

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix} = T \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} T^{-1},$$

where $T$ is an invertible matrix and $w = (i\sqrt{4-\lambda^2} + \lambda)/2$. Therefore $\det B_k(\lambda) = |w|^2$ for $\lambda \in G$, and there is a constant $M > 0$ such that

$$\left\| \prod_{k=2}^{n} B_k(\lambda) \right\|^2 = \left\| T \begin{pmatrix} w^{n-1} & 0 \\ 0 & \bar{w}^{n-1} \end{pmatrix} T^{-1} \right\|^2 \leqslant M|w^{n-1}|^2 = M \prod_{k=2}^{n} \left|\det B_k(\lambda)\right|$$

for all $n \geqslant 2$. This gives $\{B_n(\lambda)\}_{n\geqslant 2} \in H$ for $\lambda \in (-2; 2)$, i.e., the assumptions of Theorem 1.2 are satisfied.

## 2. Two criterions for the $H$-class

We present here two simple criterions guaranteeing that a sequence of matrices is in the $H$-class. For a sequence $a = \{a_n\}_{n\geqslant n_0}$ of elements of a normed space $(X, \|\cdot\|)$ we denote by $\Delta a = \{(\Delta a)_n\}_{n\geqslant n_0}$ the discrete derivative of $a$, i.e., $(\Delta a)_n = a_{n+1} - a_n$ for $n \geqslant n_0$. We write $a \in l^1$ (or *$a$ is an $l^1$ sequence*) iff $\sum_{n=n_0}^{+\infty} \|a_n\| < +\infty$, and we write $a \in D^1$ (or *$a$ is a $D^1$ sequence*) iff $\Delta a \in l^1$ (we use the same symbols $\Delta, l^1, D^1$ for all $n_0$ and in different cases of complex and matrix sequences). The symbol $\to$ denotes convergence of sequences. Note that if $X$ is a Banach space, then $D^1$ sequences are convergent. Moreover, if $X$ is a Banach algebra, then the class $D^1$ is an algebra. We denote by $\operatorname{discr} C$ the discriminant of the characteristic polynomial of $2\times 2$ matrix $C$, i.e., $\operatorname{discr} C = (\operatorname{tr} C)^2 - 4\det C$.

**Criterion 1.** *If $\{C_n\}_{n\geqslant n_0}$ is a $D^1$ sequence of real $2\times 2$ invertible matrices and $C_n \to C$ with $\operatorname{discr} C < 0$, then $\{C_n\}_{n\geqslant n_0} \in H$.*

**Criterion 2.** *Suppose that $\{C_n\}_{n\geqslant n_0}$ is a sequence of complex $2\times 2$ invertible matrices, and $C_n$ has the form $C_n = a_n I + p_n S_n + R_n$, where*

(1) *$a_n, p_n \in \mathbb{R}$ for $n \geqslant n_0$, $a_n \to a \neq 0$, and $p_n \to 0$;*
(2) *$\{S_n\}_{n\geqslant n_0}$ is a $D^1$ sequence of real matrices and $S_n \to S$ with $\operatorname{discr} S < 0$;*
(3) *$\{R_n\}_{n\geqslant n_0} \in l^1$.*

*Then $\{C_n\}_{n\geqslant n_0} \in H$.*

To prove these criterions, we need the following two lemmas.

**Lemma 2.1.** *Suppose that $\{C_n\}_{n\geqslant n_0}$ is a sequence of complex $2\times 2$ invertible matrices, $C_n \to C$ with $C$—invertible, and that $C_n$ have the form $C_n = T_n \Lambda_n T_n^{-1} + R_n$, where*

(1) *$\{T_n\}_{n\geqslant n_0} \in D^1$ and $T_n \to T$ with $T$—invertible;*
(2) *$\|\Lambda_n\|^2 = |\det \Lambda_n|$ for $n \geqslant n_0$;*
(3) *$\{R_n\}_{n\geqslant n_0} \in l^1$.*

*Then $\{C_n\}_{n\geqslant n_0} \in H$.*

**Proof.** We have $\Lambda_n = T_n^{-1}(C_n - R_n)T_n \to T^{-1}CT$ and $T^{-1}CT$ is invertible; thus there is $N \geqslant n_0$ such that $\Lambda_n$ is invertible for $n \geqslant N$ and $\{\Lambda_n^{-1}\}_{n\geqslant N}$ is bounded. Hence for such $n$ we have $C_n = T_n[\Lambda_n(I + R_n')]T_n^{-1}$ with $R_n' = \Lambda_n^{-1}T_n^{-1}R_nT_n$ and $\{R_n'\}_{n\geqslant N} \in l^1$. Thus $\det(I + R_n') = 1 + r_n \neq 0$ with $\{r_n\}_{n\geqslant N} \in l^1$ and $\det \Lambda_n = (1 + s_n)\det C_n$, where $\{s_n\}_{n\geqslant N} \in l^1$, since $s_n = -r_n/(1 + r_n)$. Therefore there is $M > 0$ such that for $n \geqslant N$

$$\prod_{k=N}^{n} |\det \Lambda_k| \leqslant M \prod_{k=N}^{n} |\det C_k|,$$

and since $\{T_n\}_{n\geqslant n_0} \in D^1$, choosing $L$ large enough, we can also obtain that

$$\prod_{k=N}^{n-1}\|T_{k+1}^{-1}T_k\| = \prod_{k=N}^{n-1}\|T_{k+1}^{-1}[T_{k+1}-(T_{k+1}-T_k)]\|$$
$$\leqslant \prod_{k=N}^{n-1}\bigl(1+\|T_{k+1}^{-1}\|\|T_{k+1}-T_k\|\bigr)\leqslant L.$$

Using now $\|\Lambda_k\|^2 = |\det \Lambda_k|$ we conclude that there is $M'$ such that

$$\left\|\prod_{k=N}^{n} C_k\right\|^2 \leqslant \left\|\prod_{k=N}^{n} T_k\bigl[\Lambda_k(I+R'_k)\bigr]T_k^{-1}\right\|^2$$
$$\leqslant \|T_n\|^2\|T_N^{-1}\|^2\left(\prod_{k=N}^{n-1}\|T_{k+1}^{-1}T_k\|\right)^2\prod_{k=N}^{n}|\det \Lambda_k|\left(\prod_{k=N}^{n}\bigl(1+\|R'_k\|\bigr)\right)^2$$
$$\leqslant M'\prod_{k=N}^{n}|\det C_k|$$

for $n \geqslant N$, and the assertion of the lemma follows from Proposition 1.1(a). □

The second lemma is a discrete version of a known result (see, e.g., [3], and see [8] or [10] for the proof of this lemma).

**Lemma 2.2.** *Let $\{B_n\}_{n\geqslant n_0}$ be a sequence of $d\times d$ complex matrices, $B_n \to B$. If $\{B_n\}_{n\geqslant n_0} \in D^1$ and $B$ has $d$ different eigenvalues $\mu_1,\dots,\mu_d$, then there exists $n_1 \geqslant n_0$, a sequence of diagonal matrices $\{\Lambda_n\}_{n\geqslant n_1} \in D^1$ and of invertible matrices $\{T_n\}_{n\geqslant n_1} \in D^1$ such that $B_n = T_n\Lambda_n T_n^{-1}$ for $n \geqslant n_1$, $\Lambda_n \to \Lambda := \operatorname{diag}(\mu_1,\dots,\mu_d)$, and $T_n \to T$, where $T$ is invertible and $B = T\Lambda T^{-1}$.*

**Proof of Criterion 1.** Since $\operatorname{discr} C \neq 0$ we can use Lemma 2.2 with $B_n := C_n$. Thus the assumptions of Lemma 2.1 with $R_n \equiv 0$ and with $n_0$ large enough are satisfied. Indeed, since $\operatorname{discr} C > 0$ and $C_n$ are real, $C_n$ and thus also $\Lambda_n$ has two different mutually conjugated proper values for large $n$ and since $\Lambda_n$ is diagonal, $\|\Lambda_n\|^2 = |\det \Lambda_n|$. Now, to obtain the assertion it is sufficient to use Proposition 1.1(a). □

**Proof of Criterion 2.** We proceed similarly to the previous proof, but we apply Lemma 2.2 to $\{S_n\}_{n\geqslant n_0}$ instead of $\{C_n\}_{n\geqslant n_0}$. Consequently, for large $n$ the matrix $C_n$ can be written in the form $T_n\Lambda'_n T_n^{-1} + R_n$ with $\Lambda'_n = a_n I + p_n\Lambda_n$, and the assumptions of Lemma 2.1 with $n_0$ large enough are satisfied. Indeed, $\Lambda_n$ has two different mutually conjugated proper values, and thus the proper values of $\Lambda'_n$ are also mutually conjugated, which yields $\|\Lambda'_n\|^2 = |\det \Lambda'_n|$. □

Note that the above two criterions can be used only when $C := \lim_{n\to+\infty} C_n$ can be diagonalized. The main difference is that Criterion 1 refers to the case of "non-degenerate" $C$ (i.e., $C$ has different proper values). In Criterion 2 the limit matrix $C$ is "degenerate" (i.e., $C$ has a double proper value). It would be very useful to find a criterion for non-diagonalizable $C$, since such "Jordan box cases" often appear when we study transfer matrices for some Jacobi matrices (see, e.g., [12]). Unfortunately, such a kind of criterion seems to be a much more difficult problem.

In this paper we shall use only Criterion 2, but Criterion 1 may be used, for instance, to obtain some of the results presented in [8].

## 3. Spectral properties of $J_\alpha$ and $J_{\epsilon,\alpha}$ with $\alpha_n = n^a$

We shall apply now Criterion 2 to study Jacobi matrices $J_\alpha$ and $J_{\epsilon,\alpha}$ for $\alpha_n = n^a$, $a > 0$. For such examples the sequence of transfer matrices $\{B_n(\lambda)\}_{n\geqslant 2}$ does not satisfy the assumptions of our criterions. Thus instead of single $B_n(\lambda)$'s we shall study the products of two neighbour $B_n(\lambda)$'s.

**Proposition 3.1.** *Let $\alpha_n = n^a$ and let $\{B_n(\cdot)\}_{n\geqslant 2}$ be the sequence of transfer matrices for $J_\alpha$ or for $J_{\epsilon,\alpha}$. If $a \in (0; 1)$, then $\{B_n(\lambda)\}_{n\geqslant 2} \in H$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. If $a = 1$, then $\{B_n(\lambda)\}_{n\geqslant 2} \in H$ for all $\lambda \in \mathbb{R} \setminus [-1/2; 1/2]$.*

**Proof.** Observe first that by (1.2) the matrices $B_n(\lambda)$ are convergent to the invertible matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and thus we can apply Proposition 1.1(b). Define

$$\tilde{B}_n(\lambda) = \begin{cases} B_{2n}(\lambda)B_{2n-1}(\lambda) & \text{for } J_\alpha, \\ B_{2n+1}(\lambda)B_{2n}(\lambda) & \text{for } J_{\epsilon,\alpha} \end{cases}$$

for $n \geqslant 2$. By (1.2), for $a \in (0; 1]$ and for both $J_\alpha$ and $J_{\epsilon,\alpha}$ cases we compute

$$\tilde{B}_n(\lambda) = \begin{pmatrix} -(1-\frac{1}{n})^a & \frac{\lambda}{n^a} \\ -(1-\frac{1}{n})^a \frac{\lambda}{n^a} & -1 + \frac{\lambda^2}{n^{2a}} \end{pmatrix} = -I + \frac{1}{n^a} S_n(\lambda) + R_n(\lambda),$$

where

$$\{R_n(\lambda)\} \in l^1 \quad \text{and} \quad S_n(\lambda) = \begin{pmatrix} \frac{a}{n^{1-a}} & \lambda \\ -\lambda & \frac{\lambda^2}{n^a} \end{pmatrix} \quad \text{for any } \lambda.$$

Thus $\{S_n(\lambda)\} \in D^1$ and $S_n(\lambda) \to S(\lambda)$, where in the case $a \in (0; 1)$ we have

$$S(\lambda) = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \quad \text{and} \quad \operatorname{discr} S(\lambda) = -4\lambda^2 < 0 \quad \text{for } \lambda \neq 0.$$

In the case $a = 1$ we have

$$S(\lambda) = \begin{pmatrix} 1 & \lambda \\ -\lambda & 0 \end{pmatrix} \quad \text{and} \quad \operatorname{discr} S(\lambda) = 1 - 4\lambda^2 < 0 \quad \text{only for } |\lambda| > \frac{1}{2}.$$

Thus the assertion follows from Criterion 2. □

Using Proposition 3.1 and Theorem 1.2 we obtain quite strong spectral information for $a \in (0; 1]$ (especially about the absolutely continuous part of the spectra). To obtain more information and to study $a > 1$ we need some extra results. Below we always assume that $\alpha$ is a positive sequence and $\epsilon > 0$.

**Proposition 3.2.** $0 \notin \sigma_{\mathrm{p}}(J_\alpha)$; *moreover,* $0 \in \sigma_{\mathrm{p}}(J_{\epsilon,\alpha})$ *iff* $\sum_{n=1}^{+\infty}(1/\alpha_n^2) < +\infty$.

The above follows immediately from Janas and Naboko criterion for '$0 \in \sigma_{\mathrm{p}}(J)$' (see [11]), which has been easily obtained by the direct computation of the possible proper vector for 0. It has been also proved in [5] by the use of other methods.

**Proposition 3.3.** $\sigma_{\mathrm{ess}}(J_{\epsilon,\alpha}) = \sigma_{\mathrm{ess}}(J_\alpha)$.

**Proof.** Let $0_1$ be the zero operator in $\operatorname{lin}\{e_1\}$. Let $J'_\alpha$ be the operator in $\operatorname{lin}\{e_1\}^\perp$ given by $J'_\alpha := UJ_\alpha U^{-1}$, where $U : l^2(\mathbb{N}) \to \operatorname{lin}\{e_1\}^\perp$ is the unitary operator determined by $Ue_n = e_{n+1}$ for $n \in \mathbb{N}$. We have $J_{\epsilon,\alpha} = (0_1 \oplus J'_\alpha) + R_\epsilon$, where $R_\epsilon$ is a two-dimensional bounded operator, and so, using Weyl theorem and the fact that $\operatorname{lin}\{e_1\}$ has finite dimension, we obtain $\sigma_{\mathrm{ess}}(J_{\epsilon,\alpha}) = \sigma_{\mathrm{ess}}(0_1 \oplus J'_\alpha) = \sigma_{\mathrm{ess}}(J'_\alpha) = \sigma_{\mathrm{ess}}(J_\alpha)$. $\square$

Before we formulate the last proposition, let us state some more notations, which will be also used in Section 4. The space of all complex sequences we denote by $l(\mathbb{N})$, $l^2_{\mathrm{e}}(\mathbb{N}) = \overline{\operatorname{lin}\{e_n\colon n \text{ is even}\}}$ and $l^2_{\mathrm{o}}(\mathbb{N}) = l^2_{\mathrm{e}}(\mathbb{N})^\perp = \overline{\operatorname{lin}\{e_n\colon n \text{ is odd}\}}$. Thus we have $l^2(\mathbb{N}) = l^2_{\mathrm{e}}(\mathbb{N}) \oplus l^2_{\mathrm{o}}(\mathbb{N})$. For $u \in l^2(\mathbb{N})$ let $u_{\mathrm{e}}$, $u_{\mathrm{o}}$ be the orthogonal projection of $u$ onto $l^2_{\mathrm{e}}(\mathbb{N})$ or onto $l^2_{\mathrm{o}}(\mathbb{N})$, respectively. Define also $U_{\mathrm{e}} : l^2(\mathbb{N}) \to l^2_{\mathrm{e}}(\mathbb{N})$ and $U_{\mathrm{o}}\colon l^2(\mathbb{N}) \to l^2_{\mathrm{o}}(\mathbb{N})$ to be the unitary transformations determined by $U_{\mathrm{e}}e_n = e_{2n}$, $U_{\mathrm{o}}e_n = e_{2n-1}$ for $n \in \mathbb{N}$. The transformation $\mathcal{C}_\alpha : l(\mathbb{N}) \to l(\mathbb{N})$ is given by $(\mathcal{C}_\alpha u)_n = (1/\alpha_n)\sum_{k=1}^n u_k$ for $u \in l(\mathbb{N})$, and $C_\alpha = \mathcal{C}_\alpha | l^2(\mathbb{N})$. Finally, let $Y$ be the unitary transformation of $l^2(\mathbb{N})$ given by the formula $(Yu)_n = (-1)^n u_n$ for $u \in l^2(\mathbb{N})$. In this paper *invertible* operator has to be surjective.

**Proposition 3.4.** *Operator $J_\alpha$ is invertible iff* $\operatorname{ran} C_\alpha \subset l^2(\mathbb{N})$. *Moreover, if* $\operatorname{ran} C_\alpha \subset l^2(\mathbb{N})$, *then $J_\alpha^{-1}$ is given by the formula*

$$J_\alpha^{-1}u = C'_\alpha u_{\mathrm{o}} + (C'_\alpha)^* u_{\mathrm{e}}, \quad u \in l^2(\mathbb{N}), \tag{3.1}$$

*where* $C'_\alpha : l^2_{\mathrm{o}}(\mathbb{N}) \to l^2_{\mathrm{e}}(\mathbb{N})$, $C'_\alpha = U_{\mathrm{e}} Y C_\alpha Y U_{\mathrm{o}}^{-1}$.

**Proof.** Let $J = J_\alpha$. Observe first that by (0.1) and since $q_n \equiv 0$ we have $u \in D(J)$ iff $u_{\mathrm{e}}, u_{\mathrm{o}} \in D(J)$, and moreover, if $u \in D(J)$ then $Ju_{\mathrm{e}} \in l^2_{\mathrm{o}}(\mathbb{N})$ and $Ju_{\mathrm{o}} \in l^2_{\mathrm{e}}(\mathbb{N})$. Thus let $D_{\mathrm{e}} = D(J) \cap l^2_{\mathrm{e}}(\mathbb{N})$, $D_{\mathrm{o}} = D(J) \cap l^2_{\mathrm{o}}(\mathbb{N})$, let $J_{\mathrm{e}}$ be the operator from $l^2_{\mathrm{e}}(\mathbb{N})$ into $l^2_{\mathrm{o}}(\mathbb{N})$ with the domain $D_{\mathrm{e}}$, $J_{\mathrm{e}} = J|D_{\mathrm{e}}$, and let $J_{\mathrm{o}}$ be the operator from $l^2_{\mathrm{o}}(\mathbb{N})$ into $l^2_{\mathrm{e}}(\mathbb{N})$ with the domain $D_{\mathrm{o}}$, $J_{\mathrm{o}} = J|D_{\mathrm{o}}$. It follows that $Ju = J_{\mathrm{e}}u_{\mathrm{e}} + J_{\mathrm{o}}u_{\mathrm{o}}$ for $u \in D(J)$. Therefore, since $J_\alpha^* = J_\alpha$ (see, e.g., [15]), we easily obtain $J_{\mathrm{e}}^* = J_{\mathrm{o}}$, $J_{\mathrm{o}}^* = J_{\mathrm{e}}$. Moreover, $J$ is invertible iff $J_{\mathrm{e}}$ and $J_{\mathrm{o}}$ are invertible. Combining the last two facts we see that $J$ is invertible iff $J_{\mathrm{e}}$ is invertible iff $J_{\mathrm{o}}$ is invertible, and when $J$ is invertible, then $(J_{\mathrm{o}}^{-1})^* = J_{\mathrm{e}}^{-1}$.

So, it suffices to study the invertibility of $J_{\mathrm{e}}$. It will be more convenient to study $\tilde{J}_{\mathrm{e}} = U_{\mathrm{o}}^{-1} J_{\mathrm{e}} U_{\mathrm{e}}$ being the operator acting in $l^2(\mathbb{N})$. Using (0.1) and the definition of $D(J)$ (see the Introduction) we obtain $D(\tilde{J}_{\mathrm{e}}) = \{u \in l^2(\mathbb{N})\colon \mathcal{J}_{\mathrm{e}}u \in l^2(\mathbb{N})\}$ and $\tilde{J}_{\mathrm{e}}u = \mathcal{J}_{\mathrm{e}}u$ for $u \in D(\tilde{J}_{\mathrm{e}})$, where $(\mathcal{J}_{\mathrm{e}}u)_n = \alpha_{n-1}u_{n-1} + \alpha_n u_n$ for any $u \in l(\mathbb{N})$, $n \in \mathbb{N}$, with the convention that $\alpha_0 = u_0 = 0$. Consider now the equation $\mathcal{J}_e u = v$, where $u, v \in l(\mathbb{N})$. This equation can be easily solved, because it can be written in the equivalent form

$$u_n = \frac{(-1)^n}{\alpha_n} \sum_{k=1}^{n} (-1)^k v_k, \quad n \in \mathbb{N}.$$

Hence $J$ is invertible iff $\tilde{J}_e$ is invertible iff $\operatorname{ran} C_\alpha \subset l^2(\mathbb{N})$, and if $\operatorname{ran} C_\alpha \subset l^2(\mathbb{N})$ then $\tilde{J}_e^{-1} = Y C_\alpha Y$, and hence (3.1) holds. □

We are ready now to formulate and prove the main theorem of this paper.

**Theorem 3.1.** *Let $\alpha_n = n^a$ and $\epsilon > 0$.*

(i) *If $0 < a \leqslant 1/2$, then both $J_\alpha$ and $J_{\epsilon,\alpha}$ are absolutely continuous and $\sigma(J_\alpha) = \sigma(J_{\epsilon,\alpha}) = \mathbb{R}$.*
(ii) *If $1/2 < a < 1$, then $J_\alpha$ is absolutely continuous and $\sigma(J_\alpha) = \mathbb{R}$; $J_{\epsilon,\alpha}$ is absolutely continuous in $\mathbb{R} \setminus \{0\}$ and $\sigma_{ac}(J_{\epsilon,\alpha}) = \mathbb{R}$, $\sigma_p(J_{\epsilon,\alpha}) = \{0\}$.*
(iii) (“*The spectral gap case*”) *If $a = 1$, then $J_\alpha$ is absolutely continuous and $\sigma(J_\alpha) = \mathbb{R} \setminus (-1/2; 1/2)$; $J_{\epsilon,\alpha}$ is absolutely continuous in $\mathbb{R} \setminus [-1/2; 1/2]$, $\sigma_{ac}(J_{\epsilon,\alpha}) = \mathbb{R} \setminus (-1/2; 1/2)$, $0 \in \sigma_p(J_{\epsilon,\alpha})$, and $J_{\epsilon,\alpha}$ is discrete in $(-1/2; 1/2)$.*
(iv) *If $a > 1$, then both $J_\alpha$ and $J_{\epsilon,\alpha}$ are discrete, $0 \notin \sigma(J_\alpha)$ and $0 \in \sigma(J_{\epsilon,\alpha})$.*

**Proof.** By [15] $J_\alpha$ and $J_{\epsilon,\alpha}$ are self-adjoint for any $a > 0$. Suppose that $a \in (0; 1)$. By Proposition 3.1 and Theorem 1.2, $J_\alpha$ and $J_{\epsilon,\alpha}$ are absolutely continuous in $\mathbb{R} \setminus \{0\}$ and $\sigma_{ac}(J_\alpha) = \sigma_{ac}(J_{\epsilon,\alpha}) = \mathbb{R}$. Thus, using Proposition 3.2 we obtain both cases (i) and (ii).

If $a = 1$ then Proposition 3.1 and Theorem 1.2 give the absolute continuity of $J_\alpha$ and $J_{\epsilon,\alpha}$ in $\mathbb{R} \setminus [-1/2; 1/2]$ and the inclusions $\sigma_{ac}(J_\alpha) \supset \mathbb{R} \setminus (-1/2; 1/2) \subset \sigma_{ac}(J_{\epsilon,\alpha})$. Moreover, by Proposition 3.4, $J_\alpha$ is invertible, since in this case $C_\alpha$ is the well-known Cesaro operator. By the Hardy inequality $\|C_\alpha\| = 2$ (see, e.g., [4] for a nice proof of this fact), hence, by (3.1), we have $\|J_\alpha^{-1}\| = 2$, and thus $\sigma(J_\alpha) \subset \mathbb{R} \setminus (-1/2; 1/2)$. Therefore $\sigma(J_\alpha) = \mathbb{R} \setminus (-1/2; 1/2)$ and $J_\alpha$ is absolutely continuous also in $\mathbb{R} \setminus \{\pm 1/2\}$ (since $E_{J_\alpha}((-1/2; 1/2)) = 0$). To obtain our assertion for $J_\alpha$ in this case it would be enough to prove that $\pm 1/2$ are not proper values of $J_\alpha$. The methods presented here do not work in this point, but we can use the results of [5], which give the absolute continuity of $J_\alpha$. Now, to obtain our claim for $J_{\epsilon,\alpha}$, it suffices to use Propositions 3.2 and 3.3.

Assume now that $a > 0$. Observe that in this case $C_\alpha$ can be written in the form $C_\alpha = K_\alpha C$, where $C$ is the Cesaro operator in $l^2(\mathbb{N})$ and $K_\alpha$ is the operator of multiplication by the sequence $\{n^{1-a}\}$ in $l^2(\mathbb{N})$. Thus $K_\alpha$ and $C_\alpha$ are compact operators, and by Proposition 3.4 $J_\alpha^{-1}$ is also compact. This proves that $J_\alpha$ is discrete. To finish the proof of this case we have to use Propositions 3.2 and 3.3. □

**Remarks.** (i) The results of Theorem 3.1 can be extended on some cases of other $\alpha$, being in some sense perturbations of the considered case $\alpha_n = n^a$. For instance, it can be easily checked that the assertions of this theorem remains valid if we assume that $\alpha_n = (n + c)^a$ with $c > -1$ for cases (i), (ii), and (iv), and with the stronger assumption

$c \geqslant 0$ for case (iii). The only important difference in the proof is the following: in case (iii) we do not know whether $\|C_\alpha\| = 2$, but since $C_\alpha = L_\alpha C$, where $L_\alpha$ is the operator of multiplication by the sequence $\{n/(n+c)\}$ in $l^2(\mathbb{N})$, we have $\|C_\alpha\| \leqslant 2$, which is enough to obtain $\sigma(J_\alpha) \subset \mathbb{R} \setminus (-1/2; 1/2)$, and finally $\sigma(J_\alpha) = \mathbb{R} \setminus (-1/2; 1/2)$ (observe that this proves that in fact $\|C_\alpha\| = 2$, since $\|C_\alpha\| = \|J_\alpha^{-1}\|$). Note that the theorem of [5] on absolute continuity of $J_\alpha$ is valid also in this case.

(ii) As follows from the proof and from Theorem 1.2, assertions (i)–(iii) can be slightly improved by adding the information, that the absolutely continuous parts of $J_\alpha$ and $J_{\epsilon,\alpha}$ are unitary equivalent to $\mathbf{X}_{\mathbb{R}}$ for cases (i) and (ii), and to $\mathbf{X}_{\mathbb{R}\setminus(-1/2;1/2)}$ for case (iii).

(iii) In [5] it is shown that $-J \sim J$, if $q_n \equiv 0$. Hence the spectrum of $J_\alpha$ is symmetric with respect to 0.

## 4. The Jacobi matrices for birth and death process

In [12] Janas and Naboko have presented the detailed analysis of the Jacobi matrices related to some birth and death process (see also [14]). Here we denote these operators by $K_a$, thus $K_a$ is the Jacobi matrix with the weight sequence $\{n+a\}_{n\geqslant 1}$ and with the diagonal sequence $\{-2(n+a)\}_{n\geqslant 1}$. We consider here $a > 0$ (note that in [12] also the cases $a \leqslant 0$ are considered). The main results of [12] are the proof of the absolute continuity of $K_a$ (for $a \geqslant 0$) and a description of the spectrum of $K_a$. The method used there is based on the non-trivial analysis of the asymptotic behaviour of the formal proper solutions for $K_a$. The main difficulty which had to be omitted there was the appearance of the Jordan box for the limit of transfer matrices for $K_a$ (being also the serious problem from the point of view of the methods presented in this paper). It turns out that the case $a > 0$ can be easily studied by the use of our results from Section 3. The above difficulty does not appear here in fact, since we do not analyze the operators $K_a$ directly, but we use a trick from [5] with the decomposition of the square of a Jacobi matrix with zero diagonal. We also use the following lemma being a generalization of an idea from [5].

**Lemma 4.1.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function and suppose that $J, K, L$ are self-adjoint operators in $l^2(\mathbb{N})$ such that $J$ and $L$ are Jacobi matrices, and*

$$g(J) = \left(U_e K U_e^{-1}\right) \oplus \left(U_o L U_o^{-1}\right).$$

*Then $E_{L,e_1}(\omega) = E_{J,e_1}(g^{-1}(\omega))$ for any Borel $\omega \subset \mathbb{R}$. Moreover, if $g$ is continuous, then $\sigma(L) = \overline{g(\sigma(J))}$.*

**Proof.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function. Since $U_o e_1 = e_1$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \phi\, dE_{L,e_1} &= \left(\phi(L)e_1, e_1\right) = \left(\phi(L)U_o^{-1}e_1, U_o^{-1}e_1\right) = \left(\phi\left(U_o L U_o^{-1}\right)e_1, e_1\right) \\
&= \left(\phi\left(U_o L U_o^{-1}\right)e_1 + \phi\left(U_e K U_e^{-1}\right)0, e_1\right) = \left(\phi\left(g(J)\right)e_1, e_1\right) \\
&= \int_{\mathbb{R}} (\phi \circ g)\, dE_{J,e_1}.
\end{aligned}$$

Thus, taking $\phi = \chi_\omega$ we obtain the first assertion of the lemma. This proves also the second assertion, because $e_1$ is a cyclic vector for any self-adjoint Jacobi matrix $M$, and thus $\sigma(M) = \operatorname{supp} E_{M,e_1}$. $\square$

Now we have all the tools sufficient to prove the following result.

**Theorem 4.1.** *For any $a > 0$ the operator $K_a$ is absolutely continuous and $\sigma(K_a) = (-\infty; -1]$.*

**Proof.** For a positive sequence $\alpha = \{\alpha_n\}$ and $\epsilon > 0$ let $\tilde{J}_\alpha$ be the Jacobi matrix with $\lambda_{2n-1} = -\lambda_{2n} = \alpha_n$, $q_n = 0$, and let $\tilde{J}_{\epsilon,\alpha}$ be the Jacobi matrix with $\lambda_1 = \epsilon$, $-\lambda_{2n} = \lambda_{2n+1} = \alpha_n$, $q_n = 0$ for $n \in \mathbb{N}$. Let us fix $\alpha_n = \sqrt{n+a}$, $\epsilon = \sqrt{a}$ and define Jacobi matrices $L_a$, $K'_a$, $L'_a$ as follows:

| Operator | $\lambda_n :=$ | $q_n :=$ |
|---|---|---|
| $L_a$ | $\sqrt{(n+a-1)(n+a)}$ | $\begin{cases} -a-1 & \text{for } n=1, \\ -2(n+a)+1 & \text{for } n>1 \end{cases}$ |
| $K'_a$ | $\sqrt{(n+a+1)(n+a)}$ | $-2(n+a)-1$ |
| $L'_a$ | $n+a$ | $\begin{cases} -a-2 & \text{for } n=1, \\ -2(n+a) & \text{for } n>1 \end{cases}$ |

The weight sequences of all the operators $K_a$, $L_a$, $K'_a$, $L'_a$ satisfy the Carleman condition $\sum_{n=1}^{+\infty}(1/|\lambda_n|) = +\infty$ and thus these operators are self-adjoint and essentially self-adjoint on the domain $\operatorname{lin}\{e_n\colon n \in \mathbb{N}\}$ (see, e.g., [2]). Observe that

$$-\bigl(\tilde{J}^2_{\epsilon,\alpha} + I\bigr) = \bigl(U_{\mathrm{e}} K_a U_{\mathrm{e}}^{-1}\bigr) \oplus \bigl(U_{\mathrm{o}} L_a U_{\mathrm{o}}^{-1}\bigr), \tag{4.1}$$

$$-\bigl(\tilde{J}^2_{\alpha} + I\bigr) = \bigl(U_{\mathrm{e}} K'_a U_{\mathrm{e}}^{-1}\bigr) \oplus \bigl(U_{\mathrm{o}} L'_a U_{\mathrm{o}}^{-1}\bigr). \tag{4.2}$$

Indeed, it can be easily checked that the both equalities hold on the domain $\operatorname{lin}\{e_n\colon n \in \mathbb{N}\}$, being the essential domain for the RHS of each of them and thus, since the both sides are self-adjoint, they have to be equal. Moreover we have $\tilde{J}_\alpha \sim J_\alpha$ and $\tilde{J}_{\epsilon,\alpha} \sim J_{\epsilon,\alpha}$ (since for the zero diagonal case the Jacobi matrices having the same absolute value of the weight sequences are unitary equivalent by a diagonal unitary transformation). Hence by Remark (i) to Theorem 3.1, by (4.1), and by properties of the direct sum of operators (see, e.g., [17]) $K_a$ is absolutely continuous. Similarly, by Remark (i) to Theorem 3.1, by (4.2), and by Lemma 4.1, $\sigma(L'_a) = (-\infty; -1]$. By the definitions of $K_a$ and $L'_a$ we have $L'_a = K_a + R_a$, where $R_a$ is a one-dimensional operator, and thus using the absolute continuity of $K_a$ and the Weyl theorem we obtain $\sigma(K_a) = \sigma_{\mathrm{ess}}(K_a) = \sigma_{\mathrm{ess}}(L'_a) = (-\infty; -1]$. $\square$

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